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Has Quantum Field Theory the Standard Transition From Poisson Brackets?

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This paper is devoted to the problem of the validity of claims that commutation rules of quantum field theories have its origin in Poisson brackets of classical mechanics.

KEY WORDS: quantum mechanics; quantum field theory.

1. INTRODUCTION

In quantum field theory a physical matter particle is understood as a field that is (Teller, 1995, pp. 103–106) "field configurations *which* are assignments of values of quantities to spacetime points, where the values may be governed by field equations." In quantum field theory (nonlinear field equations are excluded) this field "will in addition assume superimposability of the field configurations. The superposition of two field configurations is again a configuration of the field." This matter field is an object, which is extended and continues in space. It has its mathematical existence in the Fock space, combining "the concepts of superposition, fields, and quanta understood as entities with discreteness and at least a high degree of localizability . . . ," and hence it is seen as a set of strictly point-like aggregated (Teller, 1995, p. 31) quanta.

This quantum field theory assumption has many consequences, which sometimes are not so easily seen. Some of them lay almost at the beginning of the construction of the mathematical formalism, yet clearly inside the model. Others, like the one below, appear only in the context of the well known bridges between different fields of physics, but so close to the basis that we pass them by, going to more advanced problems. Finally, this paper is devoted to the problem of the validity of claims that commutation rules in quantum field theories have their origin in Poisson brackets of classical mechanics.

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2. VALIDITY OF THE TRANSITION

The quantization rule of quantum mechanics follows from the Heisenberg commutation relations, Poisson brackets imitators:

$$[x_i, p_j] = i\hbar\delta_{ij}, \qquad [x_i, x_j] = [p_i, p_j] = 0, \quad (i, j = 1, 2, 3), \tag{1}$$

where the momentum p_i is defined canonically as $\partial L/\partial \dot{x}_i$, $\bar{x} = (x_i)$ and $\bar{p} = (p_i)$ are the position and momentum vectors, respectively. In scalar quantum field theory, $\phi(\bar{x}, t)$ plays a role similar to the position vector $\bar{x}(t)$ in quantum mechanics. $\phi(\bar{x}, t)$ describes a system with an infinite number of degrees of freedom with an independent value of ϕ at each point in space at any given instant of time.

Hence let us start with the Lagrangian density connected with the Klein-Gordon equation for a scalar field $\phi(\bar{x}, t)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{2}$$

Because, as we said, in quantum field theory the quantity $\phi(\bar{x}, t)$ plays a role which is analogous to that played by $\bar{x}$ in quantum mechanics hence the standard procedure to obtain the quantization rule of quantum field theory is to divide the space into cells of volume δV_r, with $\phi_r(t)$ being the average value of $\phi(\bar{x}, t)$ in a cell r at time t. Similarly $\mathcal{L}_r$ is the average Lagrangian density in each cell. Then the momentum variable p_r, conjugate to ϕ_r is

$$p_r(t) = \frac{\partial L}{\partial \dot{\phi}(t)} = \delta V_r \frac{\mathcal{L}_r}{\partial \dot{\phi}_r(t)} = \delta V_r \pi_r(t), \tag{3}$$

where the field $\pi(\bar{x}, t)$ is defined by

$$\pi(\bar{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\bar{x}, t)}. \tag{4}$$

In Eq. (3) $\pi_r(t)$ is defined as the average value of $\pi(\bar{x}, t)$ in the cell r.

Now, because the Heisenberg commutation relations give

$$[\phi_r(t), p_s(t)] = i\hbar\delta_{rs} \tag{5}$$

we obtain, using Eqs. (3) and (5),

$$[\phi_r(t), \delta V_s \pi_s(t)] = i\hbar\delta_{rs}. \tag{6}$$

Until now the procedure (Ryder, 1994) was clear. But the next steps (which became a part of the standard thinking) are illegal, and lead to an error (I will recall them). Let us divide Eq. (6) by $\delta V_s \neq 0$, and we obtain the following relation:

$$[\phi_r(t), \pi_s(t)] = i\hbar \frac{\delta_{rs}}{\delta V_s}. \tag{7}$$

Then it is said (Ryder, 1994; Schiff, 1968) that Eq. (7) leads, in the limit $\delta V_s \to 0$, to the following one:

$$[\phi(\overline{x}, t), \pi(\overline{x}', t)] = i\hbar\delta(\overline{x} - \overline{x}'). \tag{8}$$

This is the standard procedure. But even if the commutation relation given by Eq. (8) implies commutation relations with the Hamiltonian that gives the correct equations of motion, yet we are not allowed to go (in the limit $\delta V_s \to 0$), from Eqs. (7) to (8). The fact is that in the limit $\lim_{\delta V_s \to 0}(i\hbar\frac{\delta_{rs}}{\delta V_s})$ (of RHS of Eq. (7)), δV_s is not allowed to be equal to 0, which might obviously be the case of Eq. (8) (with $\delta V_s = 0$ representing a point). Hence the step from Eqs. (7) to (8) can by no means be the declared step from finite (connected with Eq. (7)) to infinite number of degrees of freedom connected with Eq. (8).

But if we assume that Eq. (8) is correct (because the entire quantum field theory is based on it) and that (although in the different model but at the same time), Eq. (7) is also correct, then we should say that the Plank constant $\hbar$ includes (that it has always included) the Dirac delta function $\delta(\overline{x} - \overline{x}')$. But now the Heisenberg inequality

$$\Delta x \Delta p \geq \frac{1}{2}\hbar \equiv \text{constant} \times \delta(\Delta x) \tag{9}$$

leads to the conclusion

$$\Delta x = 0 \Rightarrow \Delta p \text{ is infinite of higher order than } \delta(0), \tag{10}$$

which means that a point particle spreads out, but every particle with size $\Delta x \neq 0$ might be stable (in the sense that it might not spread out).

3. FINAL DISCUSSION AND CONCLUSIONS

From the previous section one could clearly noticed that there is strong inconsistency in all quantum theories, in the sense that quantum mechanics with constant Planck constant is not consistent with quantum field theory where "Planck constant" is equal to constant $\times\, \delta(\overline{x} - \overline{x}')$. Or, quantum field theory has not assigned to it elegant, Poisson brackets basis, which it could have if Eq. (8) might be obtained from Eq. (7).

Moreover, we hope that transition from Poisson brackets { } to $\frac{1}{i\hbar}$[ ] is unique in the sense that it possesses a well defined transition constant $\hbar$.

Nevertheless, for $\delta V_s \neq 0 (\Delta x \neq 0)$, the whole procedure is consistent. It follows that quantum mechanics and quantum field theory are consistent (but not the same!) for $\delta V_s \neq 0$. In quantum field theory $\phi(\overline{x}, t)$ is the mathematical representation of a matter particle and we avoid inconsistency only when both $\phi(\overline{x}, t)$ and a matter particle are extended in space. Yet, for $\delta V_s = 0$ there is inconsistency and I could not say that quantum mechanics or quantum field theory is right.

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